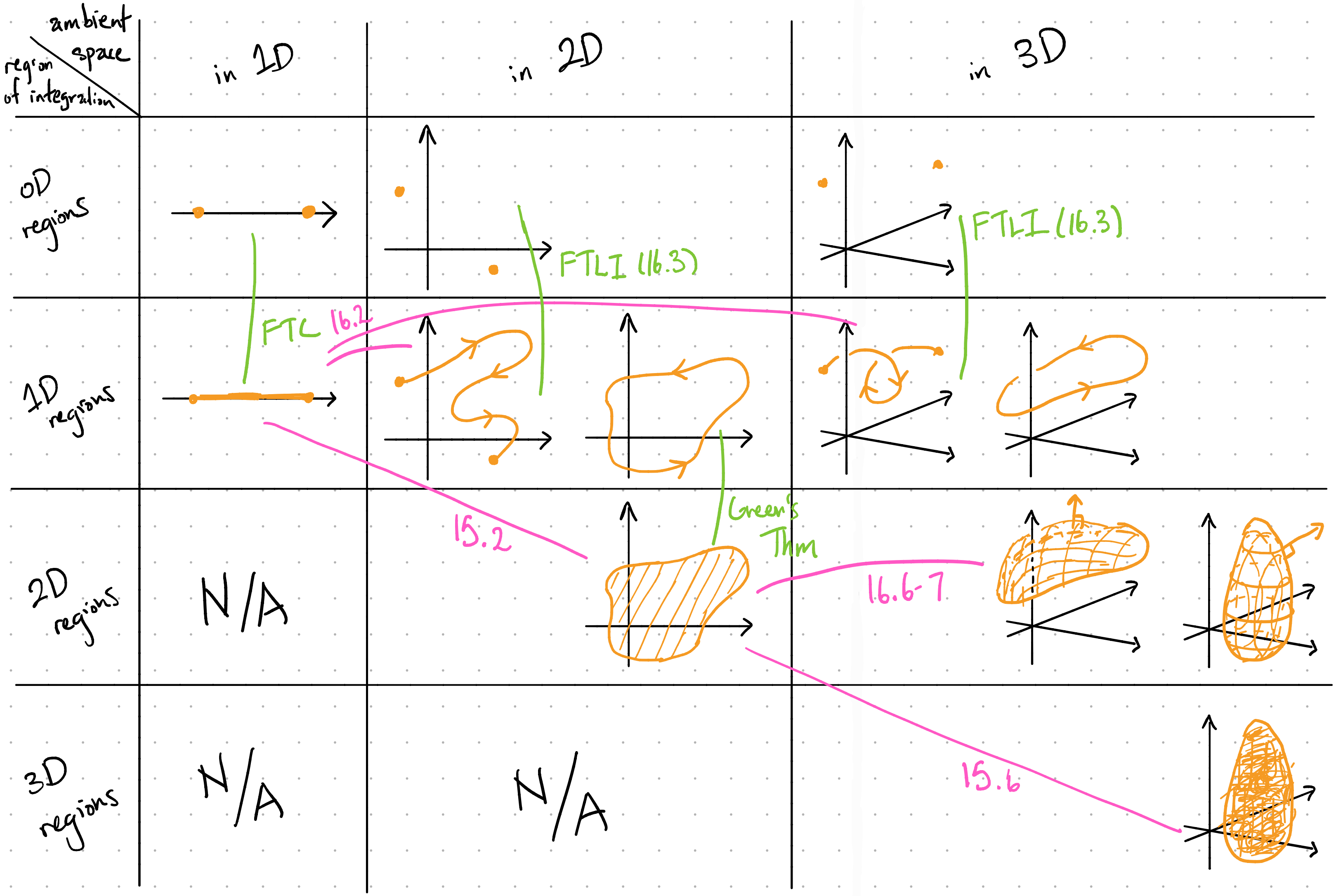
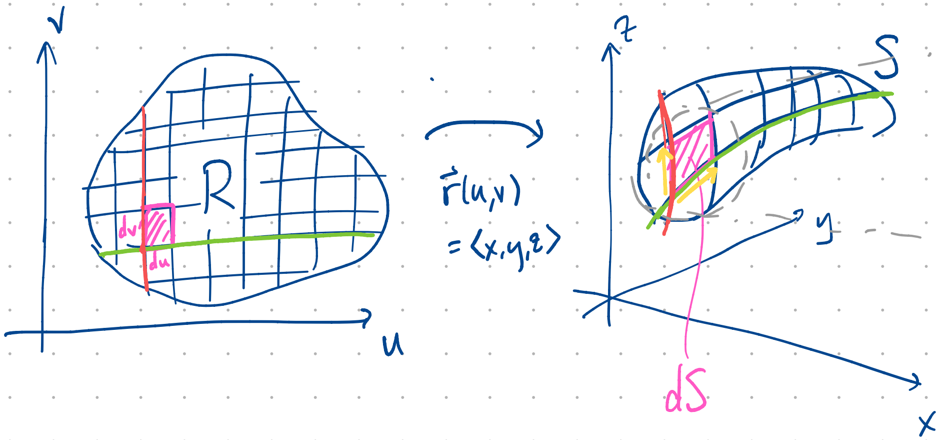


$$\int_{\text{boundary of region}} \text{thing} = \int_{\text{region}} \text{derivative}^{*} \text{ of thing}$$

* "derivative" means different things in each setting



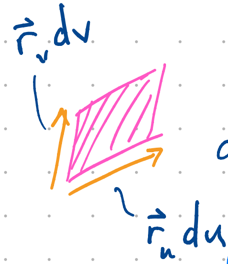
$$\iint_R f(\vec{r}(u,v))\ ??? = \iint_S f(x,y,z)\,dS$$

substitute x,y,z in terms of u,v.

how to write dS in terms of u,v?

$$\vec{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$$

= instantaneous rate of change of $\vec{r}$ with respect to u, as I hold v constant



$$dS = |\vec{r}_u\,du \times \vec{r}_v\,dv|$$

$$= |\vec{r}_u \times \vec{r}_v|\,du\,dv.$$

Let's consider applying 16.6 to two very special cases...

① S: the graph of the function $f(x,y)$ over some region R in the x,y-plane.

i.e. $S: \quad z = f(x,y)$

If we use the 'natural parametrization':

$$\begin{aligned} x &= x \\ y &= y \\ z &= f(x,y) \end{aligned} \qquad \text{i.e. } \vec{r}(x,y) = \langle x, y, f(x,y) \rangle$$

$$\vec{r}_x = \langle 1, 0, f_x \rangle$$

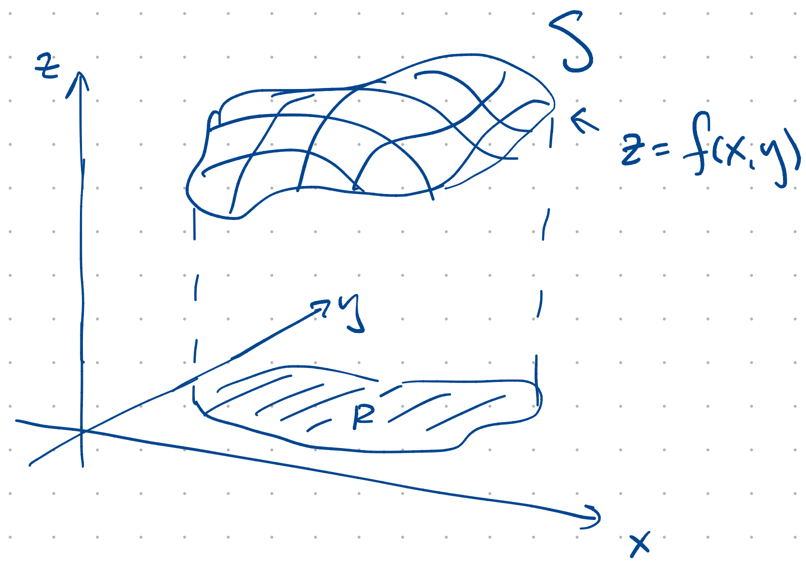
$$\vec{r}_y = \langle 0, 1, f_y \rangle$$

$$\vec{r}_x \times \vec{r}_y = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{bmatrix} = \langle -f_x, -f_y, 1 \rangle$$

So $|\vec{r}_x \times \vec{r}_y| = \sqrt{f_x^2 + f_y^2 + 1}$.

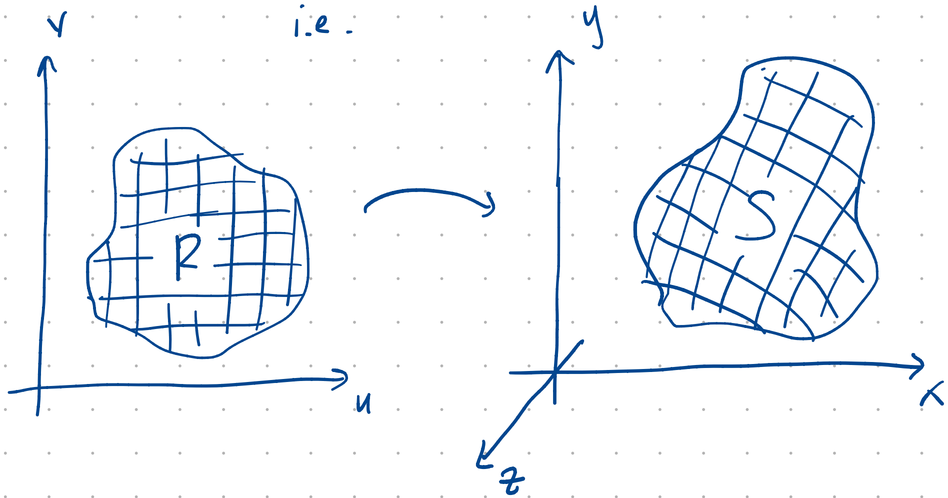
So $$\iint_S \quad dS = \iint_R \quad \sqrt{f_x^2 + f_y^2 + 1}\, dx\,dy$$

$$\boxed{\text{i.e. we recover the formula from 15.5 as a special case of 16.6.}}$$



② The surface is flat and entirely contained in the x,y-plane.

i.e.



$$\vec{r}(u,v) = \langle x, y, 0 \rangle$$

$$\vec{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, 0 \right\rangle$$

$$\vec{r}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, 0 \right\rangle$$

$$\vec{r}_u \times \vec{r}_v = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial x/\partial u & \partial y/\partial u & 0 \\ \partial x/\partial v & \partial y/\partial v & 0 \end{bmatrix} = \ldots$$

$$= \left\langle 0, 0, \underbrace{\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}}\right\rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \left| \cdots \right|$$

magnitude

absolute value.

$$= \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$$

check!

so $$\iint_S \cdots \, dx\,dy = \iint_R \cdots \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

i.e. we recover the 2D formula from 15.9 as a special case of 16.6